CH. 4
Cyclic Group

Q.1 :- The generators of Z_6 : $1, 5$.

" " " Z_8 : $1, 3, 5, 7$.

" " " Z_{20} : $1, 3, 7, 9, 11, 13, 17, 19$.

Q.2 :- The generators of $\langle a \rangle$: a^1, a^5.

" " " $\langle b \rangle$: b^1, b^3, b^5, b^7.

" " " $\langle c \rangle$: $c^1, c^3, c^7, c^9, c^{11}, c^{13}, c^{17}, c^{19}$

Q.4 :- $\langle 3 \rangle = \{3, 6, 9, 12, 15, 0\}$, $\langle a^3 \rangle = \{a^3, a^6, a^9, a^{12}, a^{15}, a^0\}$

$\langle 15 \rangle = \{15, 12, 9, 3, 6, 0\}$, $\langle a^{15} \rangle = \{a^{15}, a^{12}, a^9, a^3, a^6, a^0\}$

Q.6 :- In any Group we show. $\langle a \rangle = \langle a^{-1} \rangle$.

proof :- $a^{-1} \in \langle a \rangle \Rightarrow \langle a^{-1} \rangle \subseteq \langle a \rangle$ — (1)

$a = (a^{-1})^{-1} \in \langle a^{-1} \rangle \Rightarrow \langle a^{-1} \rangle \subseteq \langle a \rangle$ — (2)

From (1) & (2) $\langle a \rangle = \langle a^{-1} \rangle$.

Q.8 :- $|a| = 15$

a) $a^3, a^6, a^9, a^{12} \Rightarrow$ order = 5

b) $a^5, a^{10} \Rightarrow$ order = 3

c) $a^2, a^4, a^8, a^{14} \Rightarrow$ order = 15.

Q.10 :- Z_{24} , subgroup of order 8 :- $24/8 = 3$

All subgroups :- $3, 9, 15, 21$

+ a :- a^3, a^9, a^{15}, a^{21}.

$(a^3), (a^3)^3, (a^3)^5, (a^3)^7$

Q.11:— let $a^{-1} \in \langle a \rangle \Rightarrow \langle a^{-1} \rangle \subseteq \langle a \rangle$ — (1)

and $a = (a^{-1})^{-1} \in \langle a^{-1} \rangle \Rightarrow \langle a \rangle \subseteq \langle a^{-1} \rangle$ — (2)

from (1) & (2) $\Rightarrow \langle a \rangle = \langle a^{-1} \rangle$.

Q.12:— All generators : $\langle 3 \rangle$, $\langle -3 \rangle$

$\langle a^3 \rangle$: a^3, a^{-3}.

Q.13:— $\langle 21 \rangle = \{0, 21, 18, 15, 12, 9, \underline{6}, 3\}$.

$\langle 10 \rangle = \{0, 10, 20, \underline{6}, 16, 2, 12, 22, 8, 18, 4, 14\}$.

$\langle 21 \rangle \cap \langle 10 \rangle = \langle 6 \rangle$ {smallest positive multiple}.

And:— $\langle a^{21} \rangle = \{a^0, a^{21}, a^{18}, a^{15}, a^{12}, a^9, a^6, a^3\}$.

$\langle a^{10} \rangle = \{a^0, a^{10}, a^{20}, a^6, a^{16}, a^2, a^{12}, a^{22}, a^8, a^{18}, a^4, a^{14}\}$.

$\langle a^{21} \rangle \cap \langle a^{10} \rangle = \langle a^6 \rangle$.

Proof:— $\langle a^m \rangle \cap \langle a^n \rangle$, let $K = LCM(n,m)$

$K = nt_1$, $K = mt_2$

$a^K = (a^n)^{t_1} \in \langle a^n \rangle$ and

$a^K = (a^m)^{t_2} \in \langle a^m \rangle$

$\Rightarrow \langle a^K \rangle \subseteq \langle a^n \rangle \cap \langle a^m \rangle$

let $x \in \langle a^n \rangle \cap \langle a^m \rangle$

then x is multiple of $\underline{m}$ and $\underline{n}$

then x is multiple of K

So $a^x \in \langle a^K \rangle$

$a^x \in \langle a^{LCM(m,n)} \rangle$.

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Q.14:— $|G| = 49$.

$\Rightarrow$ let $|G| = n$, 7 and $n/7$ divisors of n.

If $\frac{n}{7} \neq 7$, then G has at least 4 divisors. X

So must $\frac{n}{7} = 7 \Rightarrow n = 7^2 = 49$ / In general $|G| = p^2$

Q.15:— G, abelian group, $H = \{g \in G, |g| \text{ divides } 12\}$.

One step:—

$$ab^{-1} \Rightarrow (ab^{-1})^{12} = a^{12} (b^{12})^{-1}$$

$$= e \cdot e^{-1} = e \in H$$

In general the same argument when 12 replace by any integer.

Q.16:— of Z_{240} :- $\underline{240}$

$\langle 0 \rangle \subset \langle 120 \rangle \subset \langle 60 \rangle \subset \langle 30 \rangle \subset \langle 15 \rangle \subset \langle 5 \rangle \subset \langle 1 \rangle$.

Q.18:— suppose $G=\langle g\rangle$ is cyclic of infinite order.

let g^K [illegible], $(g^K)^n = e$ {g has order dividing Kn}.

∵ (because g finite). unless $K=0$, $g^0 = e$,

⇒ Is the only element of finite order in g.

Q.27:— let complex number $Z \Rightarrow Z^n = 1$

let H subgroup $= \{Z \in C^* : Z = (1)^{\frac{1}{n}}\} = \{Z \in C^* : Z = (\cos 0 + \sin 0\, i)^{\frac{1}{n}}\}$

$= \{Z \in C^* : Z = (\cos(0 + 2K\pi) + i\sin(0 + 2\pi K))^{\frac{1}{n}},\ K = 0, 1, \ldots n-1\}$.

$= \{Z \in C^* : Z = (e^{i2K\pi})^{\frac{1}{n}},\ K = 0, 1, \ldots n-1\}$.

$= \{Z \in C^* : Z = (e^{i(\frac{2\pi}{n})})^K,\ K = 0, 1, \ldots, n-1\}$.

$= \{Z \in C^* : Z = w^K,\ K = 0, 1, \ldots, n-1,$ where $w = e^{i(\frac{2\pi}{n})}\}$.

$= \{w^0, w^1, w^2, \ldots w^{n-1}\} = \langle w \rangle$ → is the cyclic group.

and order $= n$.

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Q.28:- $\langle a^i \rangle = \langle a^j \rangle \iff i = \pm j$

if $\langle a^i \rangle = \langle a^j \rangle$ then $a^i = (a^j)^k = a^{jk}$, since a has inf order

$i = jk$ → j divides i → like wise i divides j so $i = \pm j$

$j = im$

$i = jk = imk \Rightarrow mk = 1 \Rightarrow m = k = 1 \rightarrow i = j$

$m = k = -1 \rightarrow i = -j$

Q.30:- $a, b \in G$, a odd order, $aba^{-1} = b^{-1}$, show $b^2 = e$:-

$(aba^{-1}) \Rightarrow a^2 b a^{-2} = a(aba^{-1})a^{-1} = ab^{-1}a^{-1}$

$(aba^{-1})^{-1} = (b^{-1})^{-1} = b$

So $a^2 \in C(b)$, $\langle a^2 \rangle = \langle a \rangle$, a odd order.

$\Rightarrow aba^{-1} = b$, therefore $b = b^{-1}$ and $b^2 = b.b^{-1} = e$

Q.34:- lattic of Z_8:-

1x2x4

1, 2, 4, 8

$\langle 1 \rangle$

$\langle 2 \rangle$

$\langle 4 \rangle$

$\langle 0 \rangle$

Q.40:- $\langle m \rangle \cap \langle n \rangle \Rightarrow$ The generators are LCM(m,n).

let $\langle m \rangle \cap \langle n \rangle = \langle LCM(m,n) \rangle = K$

if $a \in \langle m \rangle \cap \langle n \rangle$, the K divides a.

let $a = qK + r$, $q, r \in Z$ and $0 \le r < K$

$\Rightarrow r = a - qK$, but $K = LCM(m,n)$

So $r = 0$ and $qK = a$, K divides a.

$\Rightarrow \langle m \rangle \cap \langle n \rangle = \langle K \rangle$.

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Q.41 :- a, b are group, orders m, n, if $\langle a\rangle \cap \langle b\rangle = \{e\}$.

$\Rightarrow$ let $t = LCM(m, n)$ and $|ab| = s$, then.

$(ab)^t = a^t b^t = e$ and s divides t.

Also $e = (ab)^s = a^s b^s$ so $\rightarrow$ $\underbrace{a^s = b^{-s}}_{\text{belong to } \langle a\rangle \cap \langle b\rangle = \{e\}}$ and m divides s.

and n divides $s \rightarrow t$ divides s.

* R_{120} : rotation of $120°$, and F Reflection in D_3 :-

$\langle R_{120}\rangle \cap \langle F\rangle = \{R_0\}$, $|R_{120}| = 3$, $|F| = 2$

but D_3 its no element of order 6.

Q.53:- $\langle a\rangle \cap \langle b\rangle$ is a subgroup.

$\langle a\rangle \cap \langle b\rangle \subset \langle a\rangle$ and $\langle a\rangle \cap \langle b\rangle \subseteq \langle b\rangle$

$|\langle a\rangle \cap \langle b\rangle|$ is a common divisor of 10, 21 then

$|\langle a\rangle \cap \langle b\rangle| = 1 \longrightarrow |\langle a\rangle \cap \langle b\rangle| = \{e\}$.

Q.54:- let $x \in \langle a\rangle \cap \langle b\rangle$

$x \in \langle a\rangle \rightarrow |x| / |a|$ and $x \in \langle b\rangle \rightarrow |x| / |b|$

$|x|$ is a common divisor of 2 relatively prime integers $|a|$ and $|b|$ so it must be $|x| = 1$.

The only element with order 1 is the identity so $x = e$.

Q.55:- $|\langle a\rangle \cap \langle b\rangle|$ must divide both 24 and 10

So $|\langle a\rangle \cap \langle b\rangle| = 1$ or 2